Calculus I With Group Work
by Eric Hsu

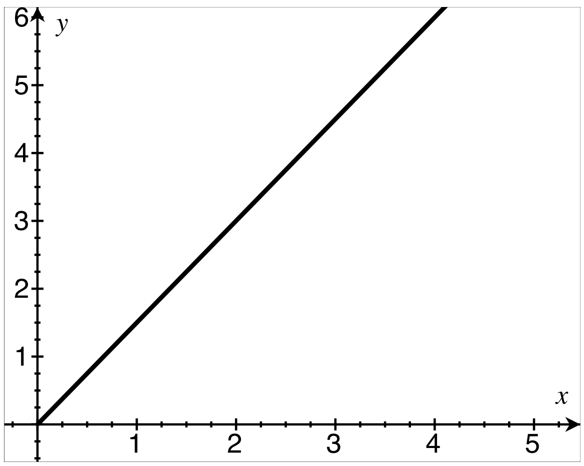
Raw Materials Piloted in Fall 2008
at San Francisco State University

Usage Notes are at `http://math.sfsu.edu/hsu`

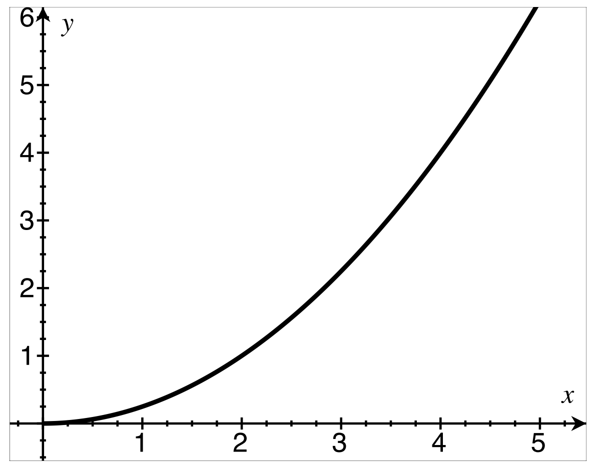
Math 226 Worksheet 1

1. At summer camp, a child comes out every morning to raise a flag. Consider the height of the flag as a function of time. Sketch what such a graph might look like.

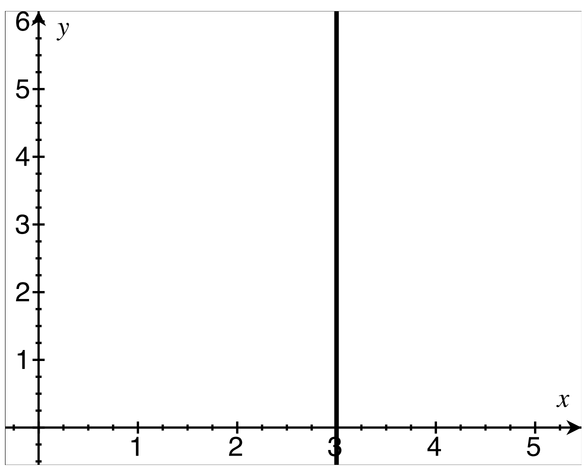
2. Consider these candidates for the graph in (1). Explain what each graph would mean. What would reasonable units be for each axis? Which one seems the most realistic to you? The least realistic?



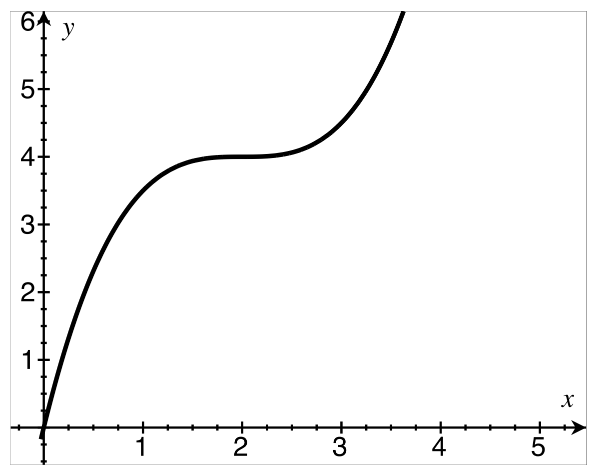
(A)



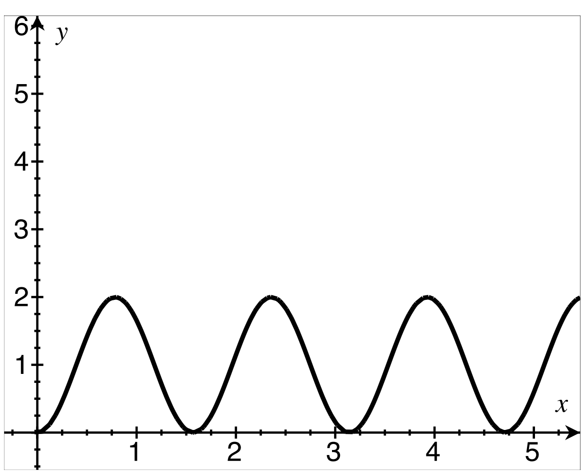
(B)



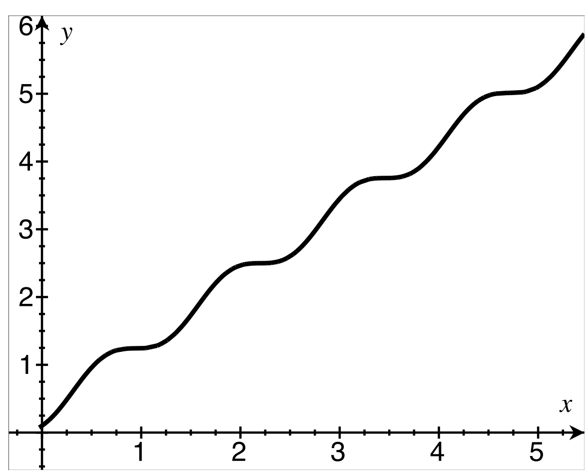
(C)



(D)



(E)



(F)

Math 226 Worksheet 2

1. Without using a graphing calculator, try to match the following six equations to the six graphs in Worksheet 1:

$$y = -\frac{1}{5}\cos(5x) + \frac{3}{10} + x; \qquad y = \frac{1}{2}(x-2)^3 + 4; \qquad x = 3;$$

$$y = \frac{3}{2}x; \qquad y = -\cos(4x) + 1; \qquad y = (\frac{x}{2})^2;$$

2. There is a speed trap in my hometown. It is a straight stretch of road and at the start, the speed limit suddenly drops to 60 miles per hour. There is a machine that makes a graph for each car that drives on this road, graphing the position of the car (in miles) as a function of time (in <u>minutes</u>).

   Suppose the six graphs from Worksheet 1 are graphs that this machine recorded.

   (a) Did Car (A) break the speed limit? What was its velocity, and at which times?

   (b) Which of the other cars broke the speed limit? Tell me when the violations happened.

   (c) For each car, pick a specific time and estimate as well as you can its velocity at that moment. You can use a calculator for this part.

   (d) Somewhere near the 4 mile mark, there is a stop sign. Did any cars actually stop for it?

Math 226 Worksheet 2

1. Without using a graphing calculator, try to match the following six equations to the six graphs in Worksheet 1:

$$y = -\frac{1}{5}\cos(5x) + \frac{3}{10} + x; \qquad y = \frac{1}{2}(x-2)^3 + 4; \qquad x = 3;$$

$$y = \frac{3}{2}x; \qquad y = -\cos(4x) + 1; \qquad y = (\frac{x}{2})^2;$$

2. There is a speed trap in my hometown. It is a straight stretch of road and at the start, the speed limit suddenly drops to 60 miles per hour. There is a machine that makes a graph for each car that drives on this road, graphing the position of the car (in miles) as a function of time (in <u>minutes</u>).

   Suppose the six graphs from Worksheet 1 are graphs that this machine recorded.

   (a) Did Car (A) break the speed limit? What was its velocity, and at which times?

   (b) Which of the other cars broke the speed limit? Tell me when the violations happened.

   (c) For each car, pick a specific time and estimate as well as you can its velocity at that moment. You can use a calculator for this part.

   (d) Somewhere near the 4 mile mark, there is a stop sign. Did any cars actually stop for it?

Math 226 Worksheet 3

1. For Cars (A), (B) and (D) in Worksheet 2, calculate the average velocity on the interval $[2,3]$, i.e. between 2 minutes and 3 minutes. Then calculate the average velocity over $[2,2.5]$. Which calculation seems closer to the instantaneous velocity at 2 minutes?

2. Your textbook suggests estimating the instantaneous velocity at 2 minutes by calculating average velocities over $[2,2+h]$, and choosing smaller and smaller values for the interval length h. Carry out this strategy for Cars (A), (B) and (D).

3. Create a graph of a position function where the average velocity over $[2,3]$ is a better estimate for the instantaneous velocity at 2 minutes than the average velocity over $[2,2.5]$.

4. For Cars (A), (B) and (D), use the strategy in (2) to estimate the instantaneous velocities at $x = 0,1,2,3$. Try to come up with a formula for the velocity function of each of the Cars. This function should take as input any time (not just $x = 0,1,2,3$) and should output the instantaneous velocity at that time. Make sure your formula fits your data and try it out on different input times.

5. Suppose a car has a position x^n miles for any time x minutes. Use your work in (4) to hypothesize what its velocity function should be.

6. For each of Cars (A), (B) and (D), write the successive approximations in (2) as a formal limit of a function. Try to evaluate the limit and prove your hypothesis in (4).

Math 226 Worksheet 3

1. For Cars (A), (B) and (D) in Worksheet 2, calculate the average velocity on the interval $[2,3]$, i.e. between 2 minutes and 3 minutes. Then calculate the average velocity over $[2,2.5]$. Which calculation seems closer to the instantaneous velocity at 2 minutes?

2. Your textbook suggests estimating the instantaneous velocity at 2 minutes by calculating average velocities over $[2,2+h]$, and choosing smaller and smaller values for the interval length h. Carry out this strategy for Cars (A), (B) and (D).

3. Create a graph of a position function where the average velocity over $[2,3]$ is a better estimate for the instantaneous velocity at 2 minutes than the average velocity over $[2,2.5]$.

4. For Cars (A), (B) and (D), use the strategy in (2) to estimate the instantaneous velocities at $x = 0,1,2,3$. Try to come up with a formula for the velocity function of each of the Cars. This function should take as input any time (not just $x = 0,1,2,3$) and should output the instantaneous velocity at that time. Make sure your formula fits your data and try it out on different input times.

5. Suppose a car has a position x^n miles for any time x minutes. Use your work in (4) to hypothesize what its velocity function should be.

6. For each of Cars (A), (B) and (D), write the successive approximations in (2) as a formal limit of a function. Try to evaluate the limit and prove your hypothesis in (4).

Math 226 Worksheet 4

1. The population of Calculus Island is modeled by the function $c(x) = x^2$, where the inputs are years after 2000 and the outputs are thousands of people.

   (a) Calculate the average rate of change of the population between 2002 and 2003.

   (b) Calculate the instantaneous rate of change of the population at the start of 2002.

2. Suppose Precalculus Island has a population modeled by the function $p(x) = x^2 + 3$, inputs and outputs as above.

   (a) Calculate the analogous answers to the two questions in (1).

   (b) Compare your answers for Calculus and Precalculus Islands. Explain why your answers are related in that way.

   (c) Do you think the relationship in (2b) holds for the instantaneous rate of change at ANY time x?

3. We find it tiresome to write 'the instantaneous rate of change of' a function f, so we will often use the equivalent term '**derivative of**' a function f. We will often write this as f'. Notice the accent. (Sometimes it is also written as $\frac{df}{dx}$.)

   (a) What is the derivative of the population function of Calculus Island at the start of 2002?

   (b) What is the derivative of $c(x)$ at $x = 2$?

   (c) What is $p'(2)$?

   (d) Write an equation relating $c'(2)$ and $p'(2)$.

   (e) Go back to that good old position function for Car B. Let's name the position function b. What are $b'(0), b'(1), b'(2)$ and $b'(3)$?

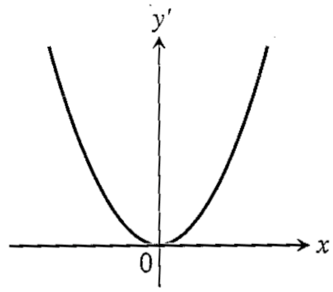
   (f) Write a formula for b' and prove your formula is true.

4. Recall that we've defined the instantaneous rate of change of a function f at input a as a limit of average rates of change over intervals $[a, a+h]$ where h tends to 0. Use this definition to write the derivative of f at input a as a formal limit.

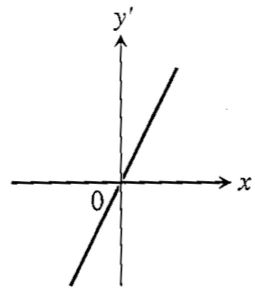
5. (bonus) Suppose $d(x) = b(x) + k$, where k is a constant that does not depend on x. Prove $d'(x) = b'(x)$.

Math 226 Worksheet 5

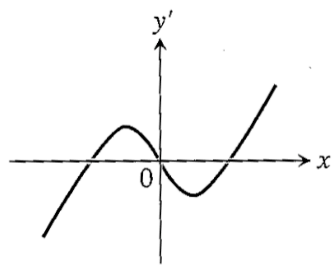
Each of the graphs (a) through (d) is the derivative of one of the graphs numbered 27-30. Match them up and explain why they match. Be able to discuss the significance of the derivative's graph, zeroes and sign.



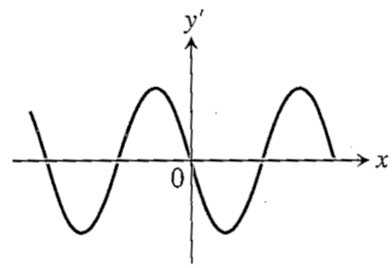
(a)



(b)

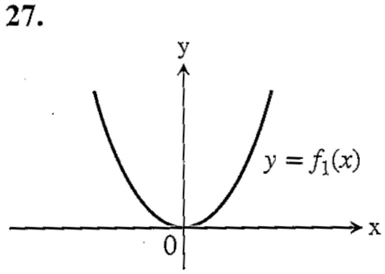


(c)

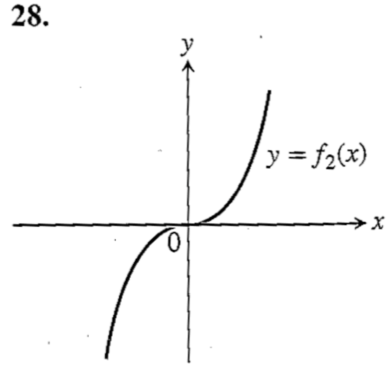


(d)

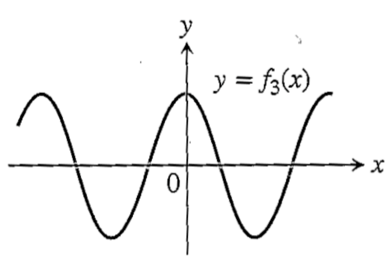
27.



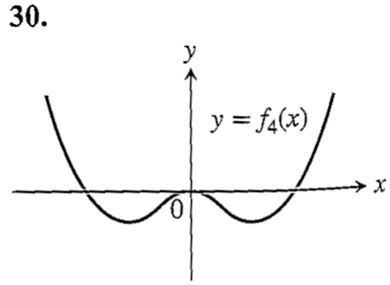
28.



29.



30.



Math 226 Worksheet 6

Haas, Section 2.3, is full of great shortcuts for calculating derivative functions. Use it as a reference for this worksheet.

1. Figure out:

$$\frac{d}{dx}(4+5), \qquad (5x^7)', \qquad \frac{d}{dt}(6+7t+8t^2), \qquad \frac{d}{dx}(6+7t+8t^2), \qquad (-\frac{1}{x^2})'$$

2. Figure out the derivative of $f(x) = x(3x-17)$ in two ways: one time using the product rule and one time by multiplying it out and finding other useful rules.

3. Consider the graph of a function f. Recall that the slope of the *tangent line* at a point $x = a$ is equal to the instantaneous rate of change at $x = a$.

   The equation of the tangent line to the graph of a function f at $x = 4$ is $y = 3x - 17$.

   (a) What is $f(4)$? What is $f'(4)$?

   (b) Given that $f(x) = ax^3 + b$, find the constants a and b.

4. Find the derivative of each of the following functions by using the product and quotient rule and then doing it with only the constant multiple, sum, difference and power rules. Do you get the same answer?

$$x\sqrt{x} \qquad \frac{1}{\sqrt[3]{x}} \qquad \frac{x^2+4x+3}{\sqrt{x}}$$

Math 226 Worksheet 6

Haas, Section 2.3, is full of great shortcuts for calculating derivative functions. Use it as a reference for this worksheet.

1. Figure out:

$$\frac{d}{dx}(4+5), \qquad (5x^7)', \qquad \frac{d}{dt}(6+7t+8t^2), \qquad \frac{d}{dx}(6+7t+8t^2), \qquad (-\frac{1}{x^2})'$$

2. Figure out the derivative of $f(x) = x(3x-17)$ in two ways: one time using the product rule and one time by multiplying it out and finding other useful rules.

3. Consider the graph of a function f. Recall that the slope of the *tangent line* at a point $x = a$ is equal to the instantaneous rate of change at $x = a$.

   The equation of the tangent line to the graph of a function f at $x = 4$ is $y = 3x - 17$.

   (a) What is $f(4)$? What is $f'(4)$?

   (b) Given that $f(x) = ax^3 + b$, find the constants a and b.

4. Find the derivative of each of the following functions by using the product and quotient rule and then doing it with only the constant multiple, sum, difference and power rules. Do you get the same answer?

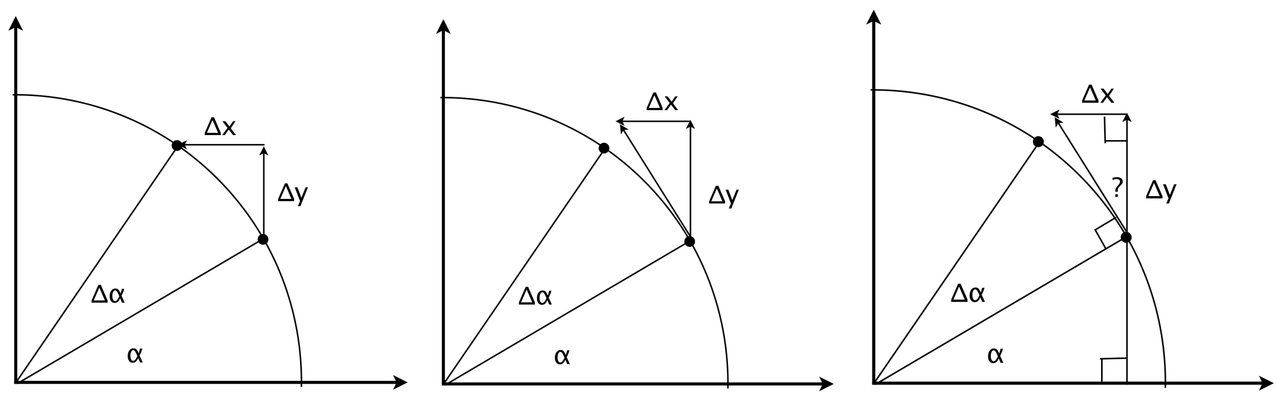
$$x\sqrt{x} \qquad \frac{1}{\sqrt[3]{x}} \qquad \frac{x^2+4x+3}{\sqrt{x}}$$

Math 226 Worksheet 7

1. Let's think about figuring out the instantaneous rate of change of the sine and cosine functions with respect to angle changes. In the left figure below, we drew a unit circle and the point α radians counter-clockwise. Remind yourself that $(\cos\alpha, \sin\alpha)$ are the coordinates of that point. Starting there, we increased the angle by some amount $\Delta\alpha$ radians. This results in a horizontal change by Δx and a vertical change by Δy as labeled in the first diagram.

   (a) Write an expression for the average rate of change in the horizontal direction between angle α and angle $\alpha + \Delta\alpha$.

   (b) Relate your answer to (1a) to the average rate of change of cosine between angle α and angle $\alpha + \Delta\alpha$



2. We could now figure out the derivative with the usual move of taking a limit of the average rate of change in (1) as $\Delta\alpha$ shrinks to 0. (In fact, the book does this... it's a messy limit.) However, we want to explore a different approach. Instead, suppose that instead of moving along the curve of the circle, the point moved along a tangent line for the same distance. The middle figure illustrates this rough approximation to the original movement. Amazingly, we now have enough information to figure out formulas for Δx and Δy, as follows.

   (a) In the third figure, we extended the vertical line down to the x-axis and we marked three angles as right angles. Convince yourself that the three angles indeed have to be right angles. You may cite correct theorems from geometry without proof.

   (b) I assured you that, by design, the length of the hypotenuse of the upper triangle equals the length of the arc that the original point travels when the angle changes by $\Delta\alpha$. What is this length?

   (c) Figure out the angle marked '?'.

   (d) Figure out Δx and Δy in terms of the mystery angle '?' and the hypotenuse length.

   (e) Figure out the average rate of change of the point's horizontal position and vertical position between angle α and angle $\alpha + \Delta\alpha$.

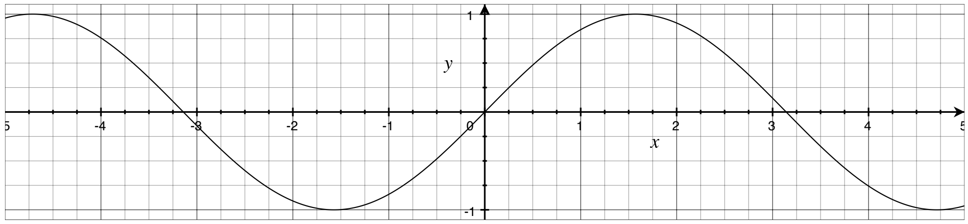
   (f) Figure out the instantaneous rate of change of the horizonal position and vertical position at angle α.

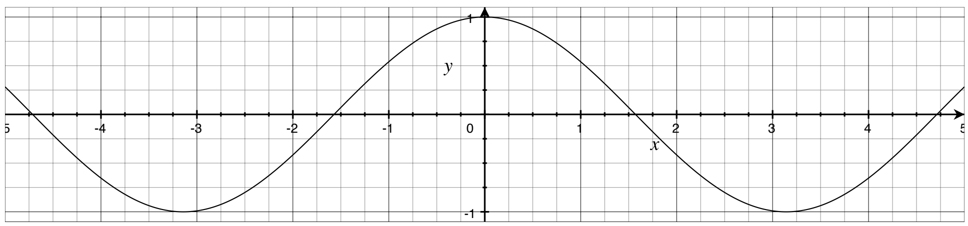
   (g) This was a rough approximation to the original movement of the point. Convince yourself that the approximation gets better and better as $\Delta\alpha$ shrinks to 0.

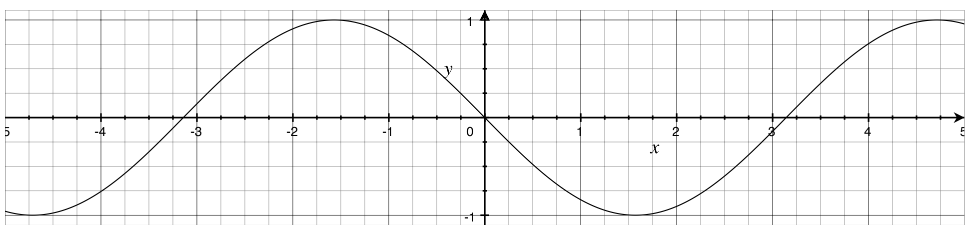
   (h) Conclude something about the derivative of sine and cosine, assuming this rough approximation becomes perfectly precise as $\Delta\alpha$ shrinks to 0.

Math 226 Worksheet 8

1. The following graphs are of the functions $\sin x, -\sin x$ and $\cos x$. Which is which?







2. Consider the graphs of $\sin x$ and $\cos x$. Convince yourself that one is the derivative of the other. Pay attention to zeroes and sign.

3. Consider the graphs of $-\sin x$ and $\cos x$. Convince yourself that one is the derivative of the other.

4. Using the quotient rule and the derivatives of $\sin x$ and $\cos x$, find the derivative of $\tan x$. Find the derivative of $\sec x$ in the same way.

5. Differentiate $(x^3)^5$ using the chain rule and check it using the power rule. Do the answers match?

6. Differentiate $f(x)/g(x) = f(x)(g(x))^{-1}$ using the product rule and the chain rule. Do you get the Quotient Rule?

Math 226 Worksheet 9

1. Differentiate the following with respect to t.

$$e^{5t}, \quad \sin(t^2), \quad (\sin(t))^2 + (\cos(t))^2, \quad \sin(e^{5t})$$

2. There is a colony of rabbits on Calculus Island. Their population for time t (years after 2000) is described by $p(t)$. The population changes in a very predictable way: if there are P rabbits at time t, then the instantaneous rate of change of the population is P rabbits added per year.

   (a) Suppose there were 25,000 rabbits at the start of 2003, what is $p'(3)$? Get the units right.

   (b) Write an equation relating p and its derivative at time t.

   (c) Try to guess a population function that fits the relationship in (2b). How many other functions can you think of that fit?

   (d) On Precalculus Island, the rabbits there breed much faster. There if there are P rabbits at time t, the instantaneous population growth rate is $10P$. Write an equation relating the population and its derivative with respect to time. Find a population function that fits this new situation.

3. Weeble Knieval is a stuntman.

   (a) Let his height above the ground at time t seconds be $h(t)$. Give meaningful interpretations to $h'(t)$ and $h''(t)$.

   (b) A simple model of gravity is that it accelerates an object towards the ground at (about) 9.8 meters per second per second, or 9.8 m/s^2. Convince yourself the units make sense.

   (c) Suppose Weeble is only experiencing acceleration due to gravity. Write an formula for h''. What is $h'(1) - h'(0)$? Get the units and sign right.

   (d) Using your knowledge of h'', guess a formula for h'. There should be an unknown constant (that doesn't change with time) in your formula. Consider time $t = 0$ to figure out your constant in terms of h.

   (e) Using your knowledge of h', guess a formula for h.

   (f) Weeble is shot out of a cannon at ground level at an initial velocity of 4.9 m/s, straight up in the air. Where is he after 1 second? Where is he at time t?

Math 226 Worksheet 10

1. Recall that the logarithm base b of x, written $\log_b x$, is defined as the number y for which $b^y = x$.

   (a) Figure out $\log_{10} 10$, $\log_{10} 1000$, $\log_{10} 1000000$, $\log_{10} 1$, $\log_{10} \frac{1}{100}$.

   (b) Without using a calculator decide the two closest integers to $\log_{10} 45$.

   (c) Using a calculator, but NOT using the $\log_{10} x$ (or $\ln x$) button, use trial and error to figure out $\log_{10} 45$ to three decimal places.

   (d) Without using a calculator decide the two closest integers to $\log_e 45$.

   (e) Using a calculator, but only using the 10^x button and NOT the $\log_{10} x$ (or $\ln x$) button, use trial and error to figure out $\log_e 45$ to three decimal places.

2. $\log_e x$ is so important that we write it with the special symbol $\ln x$ and call it the natural logarithm.

   (a) Simplify $e^{(\ln x)}$.

   (b) Suppose the derivative of the natural logarithm exists at x, i.e. $\frac{d}{dx}(\ln x)$ exists. Take the derivative of $e^{(\ln x)}$ with respect to x in two ways.

      i. Take the derivative of your answer to (2a).

      ii. Use the chain rule on $e^{(\ln x)}$. You should leave $\frac{d}{dx}(\ln x)$ in that form when it appears in your answer.

   (c) Convince yourself your answers to (2b) are equal. Then solve for $\frac{d}{dx}(\ln x)$.

3. Suppose a function f has an inverse f^{-1} whose derivative exists at x.

   (a) Simplify $f(f^{-1}(x))$.

   (b) Follow the general strategy in (2b) etc. to find a formula for the derivative of f^{-1}, $\frac{d}{dx}(f^{-1}(x))$

Math 226 Worksheet 11

1. Look up in your textbook the definitions of *local minimum/maximum/extremum*, *global minimum/maximum/extremum*, and *critical point*. For each of the following letters, draw a graph of a function f defined on all of $[0, 5]$ that

   (a) is continuous and has a local maximum that is not a global maximum.

   (b) is continuous and has a local maximum that is also a global maximum.

   (c) is continuous and has a local maximum where f' is zero.

   (d) is continuous and has a local minimum where f' is zero.

   (e) is continuous and has a point where f' is zero which is not an extremum.

   (f) is continuous and has a local maximum where f' does not exist.

   (g) is continuous and has a local minimum where f' does not exist.

   (h) is continuous and has a critical point that is not a local extremum.

   (i) is continuous and has no local extrema.

   (j) has a global maximum that is also a global minimum.

   (k) has a local maximum at $x = 3$ but no absolute maximum.

   (l) has a local maximum and min, but no absolute extrema.

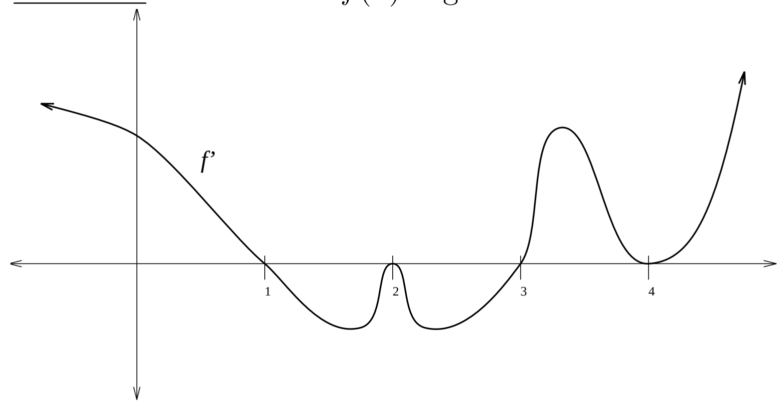
   (m) has three local minima, but no local maxima.

2. The height of a ball is expressed by $h(t) = 6t - t^2$ feet between time $t = 0$ and $t = 5$. Check the critical points and endpoints to see what the maximum and minimum height of the ball is in that time interval.

3. The population of owls on Calculus Island is modeled by the function $p(t) = t^3 - 6t^2 + 9t + 1$, where the input is in years and the output is in thousands of owls. Find the greatest population of owls between time $t = 0$ and $t = 3.5$, and when it occurs. What is the smallest population in that time interval?

Math 226 Worksheet 12

1. The graph of the <u>derivative</u> of a function $f(x)$ is given below.



   (a) What are the critical points of $f(x)$?

   (b) Make a rough sketch of $y = f(x)$. Be sure to get right the intervals where the function is increasing and decreasing, and where it levels off.

   (c) Which critical point(s) correspond to relative extrema?

2. Draw a graph of a function f that is concave down on $[0, 5]$.

   (a) What can you conclude about the derivative of f?

   (b) What can you conclude (from 2a) about the derivative of f' ?

   (c) Make an analogous argument about functions whose graphs are concave up on an interval.

3. Weeble Knieval is a stuntman.

   (a) Let his height above the ground at time t seconds be $h(t)$. Give meaningful interpretations to $h'(t)$ and $h''(t)$.

   (b) A simple model of gravity is that it accelerates an object towards the ground at (about) 9.8 meters per second per second, or 9.8 m/s^2. Convince yourself (and me) the units make sense.

   (c) Suppose Weeble is only experiencing acceleration due to gravity. Write an formula for h''. What is $h'(1) - h'(0)$? Get the units and sign right.

   (d) Using your knowledge of h'', guess a formula for h'.

   (e) Your guessed formula in (3d) is not the only possible guess, because if we add any constant to it, it will not change the derivative. Now let me give you more information. Suppose Weeble's vertical velocity at time $t = 0$ is 4.9 m/s. Find the unique formula for h' that fits the known information.

   (f) Using your knowledge of h', guess a formula for h.

   (g) Suppose Weeble began on the ground. How high is Weeble after 1 second? Where is he at time t?

   (h) Write and prove a general formula for the height of a "projectile" that experiences vertical acceleration only due to gravity. Your formula should be in terms of the initial height (at time $t = 0$) and initial velocity.

4. Suppose that g and h are increasing functions on an interval I. For the following functions, either show that they must be increasing on I or give a counter-example.

$$a)\ g + h \qquad b)\ g \cdot h \qquad c)\ g \circ h$$

Math 226 Worksheet 13b

1. State a handy test for determining whether critical points are local extrema, merely by checking the sign of the first derivative near each critical point. Why does it work? (Hint: WS12.1)
2. Look at WS12.1 and estimate $f''(a)$ for the critical points of f. What is the difference between $f''(a)$ for the local extrema and the other critical points?
3. Suppose there is an a where $f'(a) = 0$. Tell me a condition on $f''(a)$ that will ensure that there is a local extremum of f at a. (Hint: 2.)
4. Suppose $g''(a)$ is positive. What can you say about the concavity of g at a? Give a good explanation why this has to be so. What if $g''(x)$ is negative?
5. Continue 12.3 and 12.4.

Math 226 Worksheet 13b

1. State a handy test for determining whether critical points are local extrema, merely by checking the sign of the first derivative near each critical point. Why does it work? (Hint: WS12.1)
2. Look at WS12.1 and estimate $f''(a)$ for the critical points of f. What is the difference between $f''(a)$ for the local extrema and the other critical points?
3. Suppose there is an a where $f'(a) = 0$. Tell me a condition on $f''(a)$ that will ensure that there is a local extremum of f at a. (Hint: 2.)
4. Suppose $g''(a)$ is positive. What can you say about the concavity of g at a? Give a good explanation why this has to be so. What if $g''(x)$ is negative?
5. Continue 12.3 and 12.4.

Math 226 Worksheet 13b

1. State a handy test for determining whether critical points are local extrema, merely by checking the sign of the first derivative near each critical point. Why does it work? (Hint: WS12.1)
2. Look at WS12.1 and estimate $f''(a)$ for the critical points of f. What is the difference between $f''(a)$ for the local extrema and the other critical points?
3. Suppose there is an a where $f'(a) = 0$. Tell me a condition on $f''(a)$ that will ensure that there is a local extremum of f at a. (Hint: 2.)
4. Suppose $g''(a)$ is positive. What can you say about the concavity of g at a? Give a good explanation why this has to be so. What if $g''(x)$ is negative?
5. Continue 12.3 and 12.4.

Math 226 Worksheet 14

1. A rectangle has an 8 meter perimeter.
   (a) Suppose one of its sides is x meters long. Write a formula for $A(x)$, the area of the rectangle as a function of x.
   (b) What are the largest and smallest values of x that make sense as inputs?
   (c) Use critical points to find the maximum that $A(x)$ can be on its domain.
   (d) What is special about the dimensions about the optimal rectangle you found in (1c)? Conclude something in general about maximizing the area of rectangles with fixed perimeter.
2. An open-top box is to be made by cutting small congruent squares from the corners of a 12 inch by 12 inch sheet of tin and bending up the sides.
   (a) Suppose the square cut from each corner has a side of x inches. Write a formula for $V(x)$, the volume of the resulting box after cutting away a x inch square from each corner and folding up.
   (b) What is the most and least that x can be and still have the construction make sense?
   (c) Use critical points to find the maximum and minimum volume that can be constructed as described.
   (d) How large should the squares cut from the corners be to make the box hold as **much** as possible?

Math 226 Worksheet 14

1. A rectangle has an 8 meter perimeter.
   (a) Suppose one of its sides is x meters long. Write a formula for $A(x)$, the area of the rectangle as a function of x.
   (b) What are the largest and smallest values of x that make sense as inputs?
   (c) Use critical points to find the maximum that $A(x)$ can be on its domain.
   (d) What is special about the dimensions about the optimal rectangle you found in (1c)? Conclude something in general about maximizing the area of rectangles with fixed perimeter.
2. An open-top box is to be made by cutting small congruent squares from the corners of a 12 inch by 12 inch sheet of tin and bending up the sides.
   (a) Suppose the square cut from each corner has a side of x inches. Write a formula for $V(x)$, the volume of the resulting box after cutting away a x inch square from each corner and folding up.
   (b) What is the most and least that x can be and still have the construction make sense?
   (c) Use critical points to find the maximum and minimum volume that can be constructed as described.
   (d) How large should the squares cut from the corners be to make the box hold as **much** as possible?

Math 226 Worksheet 15

1. Suppose $f'(x) = 3x^2$. What could $f(x)$ be? Check your answer. Are there other possible answers?

2. Suppose $g(x) = x^5$ and $f'(x) = g(x)$. What could $f(x)$ be?

3. When $f'(x) = g(x)$ for all x in an interval I, we say f is the *antiderivative* of g on I. Find antiderivatives for the following functions.

$$x^5 - 3x^2 + 1, \qquad 2 - \frac{5}{x^2}, \qquad \frac{1}{2\sqrt[3]{x}}, \qquad \frac{2}{x}$$

4. Find antiderivatives for the following functions.

$$\sin x, \qquad \cos 5x, \qquad 6e^x, \qquad 6e^{3x},$$

5. (Hard) Find antiderivatives for the following functions.

$$2x \sin x^2, \qquad \frac{\cos(\ln x)}{x}, \qquad 6x^2 e^{x^3}$$

6. Suppose at time t hours, an owl on Calculus Island has a position of $p(t)$ miles down a straight road. Its velocity $v(t)$ is the derivative of $p(t)$. Suppose $v(t) = 1 + \sin t$.

   (a) Write all possible formulas for $p(t)$.

   (b) Suppose you know the owl was 5 miles down the road at time $t = 0$. Now how many possible formulas for $p(t)$ can you find?

   (c) Suppose instead you know that the owl was 7 miles down the road at $t = \frac{9\pi}{2}$. Now what possible formulas for $p(t)$ are there?

7. (Bonus) Suppose that g and h are increasing functions on an interval I. For the following functions, either show that they must be increasing on I or give a counter-example.

$$a)\ g + h \qquad b)\ g \cdot h \qquad c)\ g \circ h$$

Math 226 Worksheet 16a

Consider a function f whose graph is above the x-axis between $x = a$ and $x = b$. It turns out that the area of the region between the graph of $y = f(x)$, the x-axis and between $x = a$ and $x = b$ is very important to study.

This area is so important that we often just call it "the area under the graph of f between a and b". In fact, there is a strange standard notation for this area which we will practice using today: $\int_a^b f(x)dx$.

1. Draw the regions associated with the following areas. Then figure out what the areas actually are.

   (a) $\int_0^2 f(x)\,dx$ where $f(x) = 60$.

   (b) $\int_0^2 f(x)\,dx$ where $f(x) = 60x$.

   (c) $\int_2^4 f(x)\,dx$ where $f(x) = 60x$.

   (d) $\int_0^5 f(x)\,dx$ where $f(x) = \begin{cases} 3x & \text{if } x < 1, \\ 3 & \text{if } 1 \le x \le 2, \\ 5 - x & \text{if } x > 2. \end{cases}$.

2. Suppose for each case in (1), the graphed function $f(x)$ represents the **velocity** of a car along a straight track in miles per hour at time x hours.

   (a) For each case, describe in plain English what is happening to the car over the graphed time intervals.

   (b) For (1a) through (1c), make an intuitive argument calculating how far the car has moved over the time interval (i.e. its displacement). **Don't** refer to the area under the curve.

   (c) Conjecture a relationship between the area under the graph of f and the displacement of the car.

   (d) Suppose $f(x) = 30x^2$ is the velocity function. Find the displacement of the car from time 0 to 2. (Hint: antiderivative.) What should $\int_0^2 f(x)\,dx$ be according to (2c)?

   (e) (bonus) Suppose F is an antiderivative of f. What should $\int_a^b f(x)\,dx$ be according to (2c)?

3. There is a particular method people use to estimate areas under curves, called *Riemann Sums*. Let's construct a Riemann Sum with 4 subintervals for the function $f(x) = 30x^2$ on $[0, 2]$:

   (a) Divide the interval $[0, 2]$ into 4 equal subintervals.

   (b) Approximate the area above each subinterval with a rectangle whose height is equal to the value of f at the midpoint. What estimate do you get? Be sure you draw a picture of these rectangles with the graph of f.

   (c) This estimate is called the Midpoint Rule. How close is that answer to your guess from (2d)?

   (d) Divide up the interval into 8 subintervals, and get another estimate. How do you think this estimate compares with the 4 subinterval estimate? (And how close is it to your guess?)

Math 226 Worksheet 17a

1. Write the following sums in sigma notation.

   (a) $1 + 2 + \cdots + N$

   (b) $3 + 3 + 3 + 3 + 3$

   (c) $1 + 4 + 9 + 16 + 25 + ...$

2. Write out the following sigma-notated sums as full explicit sums.

   (a) $\displaystyle\sum_{i=1}^{4} \frac{1}{i+3}$

   (b) $\displaystyle\sum_{i=2}^{5} \sin x_i \cdot \Delta x$

   (c) $\displaystyle\sum_{i=-2}^{2} \frac{x^i}{i^2}$

   (d) (bonus) $\displaystyle\lim_{N\to\infty} \sum_{i=0}^{N} \frac{1}{i!}$. Use a calculator to guess the limit.

3. Write the Riemann Sums from Worksheet 16.3 in sigma notation. You should define the width of each subinterval as Δx to make your sum tidier.

4. For each equation in the list below, draw a corresponding picture involving areas under curves. Explain why the equation is true for all functions f and g, or please provide a counterexample. Here a, b, c and k are constants with respect to x, and $c \neq 0$. You may assume $a < b$.

   (a) $\int_a^a f(x)\,dx = 0$

   (b) $\int_a^b f(x)\,dx > 0$ if $a, b > 0$

   (c) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

   (d) $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

   (e) $\int_a^b c + f(x)\,dx = c + \int_a^b f(x)\,dx$

   (f) If $M < f(x) < N$ for all x in $[a, b]$, then $M(b-a) < \int_a^b f(x)\,dx < N(b-a)$

   (g) If $g(x) < f(x)$ for all x in $[a, b]$, then $\int_a^b g(x)\,dx < \int_a^b f(x)\,dx$

   (h) $\int_a^b f(x)\,dx = \int_a^{b^2} f(x)\,dx + \int_{b^2}^{b^3} f(x)\,dx + \int_{b^3}^{b} f(x)\,dx$

   (i) $\int_a^b f(x) \cdot g(x)\,dx = \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$

Math 226 Worksheet 18

1. Evaluate these definite integrals.
$$\int_1^4 \sqrt{x}\, dx, \qquad \int_{\pi/3}^{\pi/2} \sin x\, dx,$$
$$\int_1^0 e^{2t}\, dt, \qquad \int_{-8}^{8} \sqrt{64-x^2}\, dx \text{ (Hint: what shape is the graph?)}$$

2. An area under a graph is growing in the following way: at x hours, the area equals $F(x) = \int_0^x f(t)\, dt$. Here $f(t) = 60$. (Be careful about where the input x goes in the formula of F!)

   (a) Draw a picture of the areas that correspond to $F(1)$, $F(2)$, $F(0)$ and $F(x)$.

   (b) Evaluate $F(1)$, $F(2)$, $F(0)$ and $F(x)$ using (a) anti-differentiation and the FTC, and (b) geometry.

   (c) Draw a picture of the change of area on the time interval $[x, x+h]$, $F(x+h) - F(x)$ and find its value.

   (d) Write a formula for the average rate of change of $F(x)$ on the time interval $[x, x+h]$.

   (e) Use limits to figure out the instantaneous rate of change of F at x.

3. Repeat all parts of (2) for
$$G(x) = \int_0^x 2t\, dt$$

4. Suppose $F'(t) = f(t)$, where f is continuous. Use the FTC to write an expression for $\displaystyle\int_a^x f(t)\, dt$ in terms of F. Then write an expression for $\displaystyle\frac{d}{dx}\left(\int_a^x f(t)\, dt\right)$ in terms of f.

5. Compute $F'(x)$ when

   (a) $F(x) = \int_2^x \cos t\, dt$

   (b) $F(x) = \int_2^{x^2} \cos t\, dt$

   (c) $F(x) = \int_{2x}^{x^2} \cos t\, dt$

Math 226 Puzzler Worksheet

1. You bought a ten-gallon hat as a souvenir of a visit to Texas; only when you got home did you discover that the label states it to be only a six-gallon hat. By now, you were skeptical that it was even that big, and you decided to test it by trying to fill it with 6 gallon of water. The only containers you had on hand are a 9-gallon and a 4-gallon container. Using them, how were you able to pour 6 gallons into the hat?

2. 2 boys on bicycles, 20 miles apart, began racing toward each other. The instant they started, a fly on the handle bar of one of the bikes started flying toward the other bike's handle bar. As soon as it reached, it turned around and went to the other bike and so on until the bikes met. If each bike had a constant speed of 10 mph, and the fly was traveling 15 mph constantly, how far did the fly travel?

3. If a hen and a half lays an egg and a half in a day and a half, how many eggs can a hen lay in three days?

4. A census taker approaches a house and asks the woman who answers the door, "How many children do you have, and what are their ages?"
   Woman: "I have three children, the product of their ages is 36, the sum of their ages is equal to the address of the house next door."
   The census taker walks next door, comes back and says, "I need more information."
   The woman replies, "I have to go, my oldest child is sleeping upstairs."
   Census taker: "Thank you, I have everything I need." *What are the ages of the three children?*

Math 226 Puzzler Worksheet

1. You bought a ten-gallon hat as a souvenir of a visit to Texas; only when you got home did you discover that the label states it to be only a six-gallon hat. By now, you were skeptical that it was even that big, and you decided to test it by trying to fill it with 6 gallon of water. The only containers you had on hand are a 9-gallon and a 4-gallon container. Using them, how were you able to pour 6 gallons into the hat?

2. 2 boys on bicycles, 20 miles apart, began racing toward each other. The instant they started, a fly on the handle bar of one of the bikes started flying toward the other bike's handle bar. As soon as it reached, it turned around and went to the other bike and so on until the bikes met. If each bike had a constant speed of 10 mph, and the fly was traveling 15 mph constantly, how far did the fly travel?

3. If a hen and a half lays an egg and a half in a day and a half, how many eggs can a hen lay in three days?

4. A census taker approaches a house and asks the woman who answers the door, "How many children do you have, and what are their ages?"
   Woman: "I have three children, the product of their ages is 36, the sum of their ages is equal to the address of the house next door."
   The census taker walks next door, comes back and says, "I need more information."
   The woman replies, "I have to go, my oldest child is sleeping upstairs."
   Census taker: "Thank you, I have everything I need." *What are the ages of the three children?*