Solving the Cubic

This is a collection of activities around solving cubic equations that should be accessible to anyone who loves doing school algebra.

1. **Cubic Vertex Form**. We can solve any quadratic by completing the square. That is, we can write any quadratic in the vertex form $a(x-h)^2+k$. Is it always possible to write a cubic in the "cubic vertex" form $a(x-h)^3+k$ for some constants h and k ?

   (a) Rewrite x^3+3x^2+3x+9 in cubic vertex form. Use it to find one root.

   (b) Rewrite $x^3+6x^2+12x-10$ in cubic vertex form and find a root.

   (c) Rewrite $x^3+4x^2+\frac{16x}{3}x$ in cubic vertex form and find a root.

   (d) Show it is **impossible** to write x^3+3x^2+4x+1 in cubic vertex form.

   (e) *Consolation Prize*: Show how to write any cubic x^3+Bx^2+Cx+D in the form $(x-h)^3+(mx+b)$ for some constants h, m, b.

   (f) You have found that x^3+Bx^2+Cx+D equals $(x-\frac{B}{3})^3$ plus a linear function. This suggests substituting $X=x-\frac{B}{3}$. Show this produces a "Cubic-Linear Form" expression of the form X^3+pX+q, for constants p, q.

   Comment: You may have (should have!) liberally used the fact that polynomials are equal functions if and only if their corresponding coefficients are equal. If you have never proved this, consider this an assignment. It's not trivial. The cubic-linear form is more generally known as the *depressed cubic*.

2. **Cubic Linear Form**. If we knew how to solve equations of the form $X^3+pX+q=0$, then we could solve every cubic! It is not at all obvious how to do it though. The $p=0$ case is solved much like the first examples in (1), so let us only consider the harder $p \neq 0$ case.

   (a) Luckily, we only needed one person in history to find this miraculous trick and have the decency to share it! Rewrite the equation $X^3+pX+q=0$ using the substitution $X=w-\frac{p}{3w}$.

   (b) For this substitution to be meaningful, we need to know there is such a nonzero w for any X! Argue that the quadratic $w^2-wx-\frac{p}{3}$ has two complex roots and that neither one can be zero.

   (c) After some convenient cancellation, you should have something equivalent to

   $$w^3-\frac{p^3}{27w^3}+q=0$$

   That probably doesn't seem simpler, until you know the second part of the trick. Use the substitution $Y=w^3$. Show that $Y\neq 0$ and thus this equation is actually a quadratic equation in Y:

   $$Y^2+qY-(\frac{p}{3})^3=0$$

   (d) In theory, we can now solve any cubic because we know how to solve quadratics. Before we tackle a complete computation, let's practice unpacking a solution. Suppose you have already rewritten $X^3-3X-2=0$ as a quadratic equation in Y using the method of (2c). Suppose I tell you that one solution is $Y=1$. Calculate w and then one corresponding possible root X. Check that it is really a root.

Details: There are some details we glossed over here. Most importantly, the equation $w^3 = 1$ actually has three complex roots (only one is pure real). If you know a bit about complex numbers (mainly how to find the n complex nth roots of a number), you can actually calculate all the possible roots.

There is a way to motivate the trick, and when I understand that better, I may later turn it into part of this activity.

(e) Start with $X^3 - 3X - 2 = 0$. Write the corresponding quadratic equation in Y from (2c), and solve it, and then find a corresponding root X.

(f) Use this method to find a root for $X^3 - 12X - 16 = 0$. Then find the other roots by dividing out a linear term and solving the quadratic.

Bonus: Figure out all the possible values for w (i.e. all three complex cube roots) and backsolve X for each one. Check that they are really roots.

3. **The General Cubic**. Now let's put the pieces together. Take the cubic $x^3 + 9x^2 + 15x - 9$.

   (a) Use the "completing the cubic" approach from (1) to put it in cubic-linear form. You should find it is one of the cubics from (2).

   (b) Since you know the roots of the cubic-linear expression from (1), figure out the roots of the original cubic. Check that they really work!

4. Now you can, in principle, solve any cubic. Make up some of your own to try. It is challenging and educational to come up with tidy examples with tidy roots.

5. A truly patient person could take a general cubic, complete the cube, perform all substitutions, write out the general solutions to the quadratic in Y, and then backtrack through all the calculations, producing a Cubic Formula for the roots.

Final Comments. You can find various forms of the Cubic Formula online. Now that you know where it comes from, you can recognize the constituent pieces: in the inside there is always the quadratic formula (solving the quadratic in Y). This will have a cube root taken, from undoing $Y = w^3$. There will be some complexity from solving X from w. Then on the outside is always a shift by $\frac{B}{3}$ coming from undoing the initial substitution $X = x - \frac{B}{3}$.

This method is probably due to Cardano. To find a complete set of cubic roots, you will have to make sure you keep all complex roots of the quadratic in Y, and then carefully check all complex cube roots of the complex numbers. That will give you six possible roots (up to multiplicity).

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